Assignment for next week

Due: 6:30 PM Monday April 11th (2016) -> PDF by email only

- 5 points: Pose two questions based on this lecture
- 5 points: what (and why) is sinc(0)?
- 5 points: Sketch (by hand okay) rect(x,y) and sinc(x,y) and label axes
- 5 points each: Determine if these are linear functions

$$g(x) = f(x) + 1$$

$$g(x) = x f(x)$$

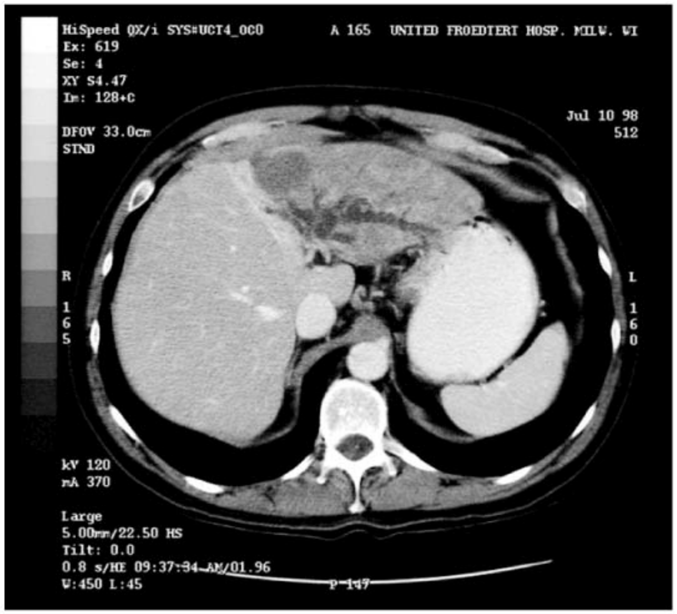
$$g(x) = \left(f(x) \right)^2$$

- 5 points: If the input to a 2D imaging system is an impulse, why is the output called a *point spread function* (PSF)

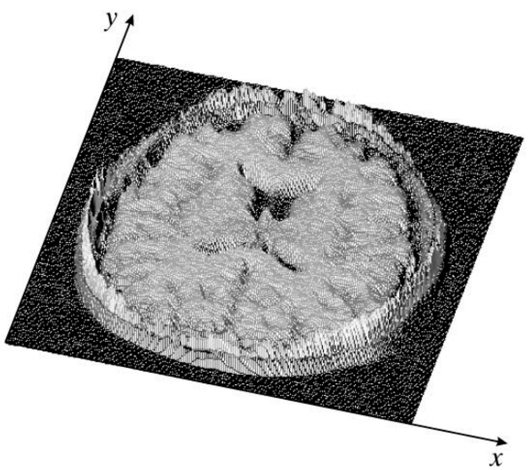
2D Signals and Systems

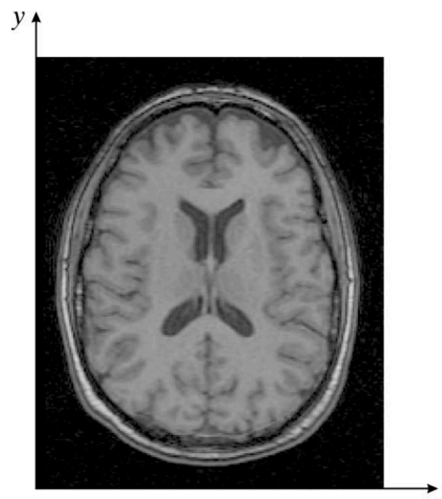
Signals

- A signal can be either continuous $f(x), f(x,y), f(x,y,z), f(\mathbf{x})$
- or discrete $f_{i,j,k}$ etc. where *i,j,k* index specific coordinates



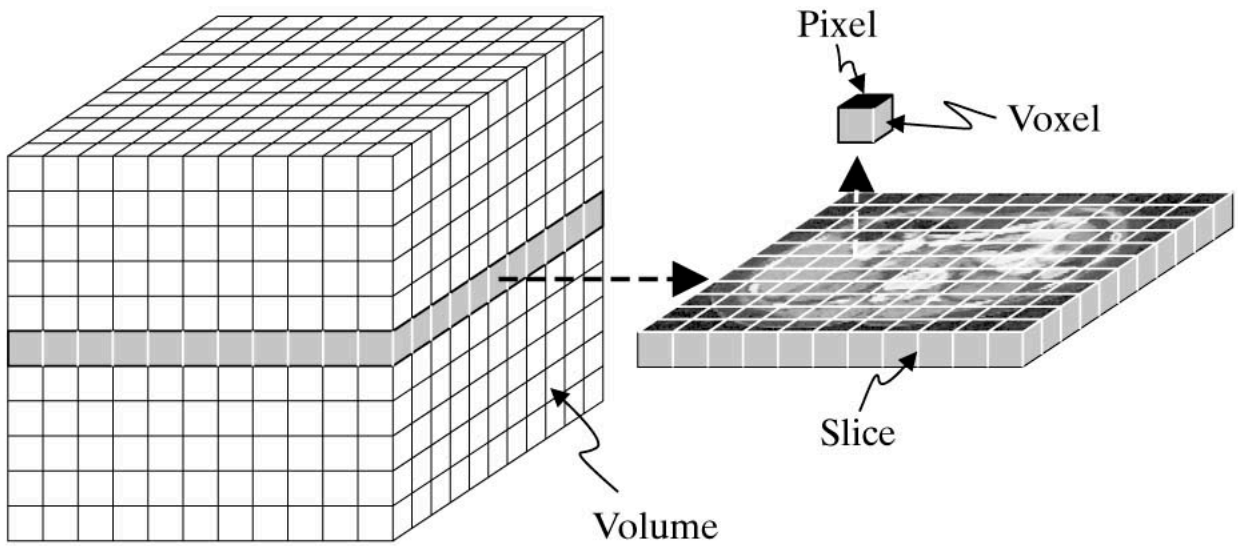
- Digital images on computers are necessarily discrete sets of data
- Each element, or bin, or voxel, represents some value, either measured or calculated





Digital Images

- Real objects are continuous (at least above the quantum level), but we represent them digitally as an approximation of the true continuous process (pixels or voxels)
- For image representation this is usually fine (we can just use smaller voxels as necessary)



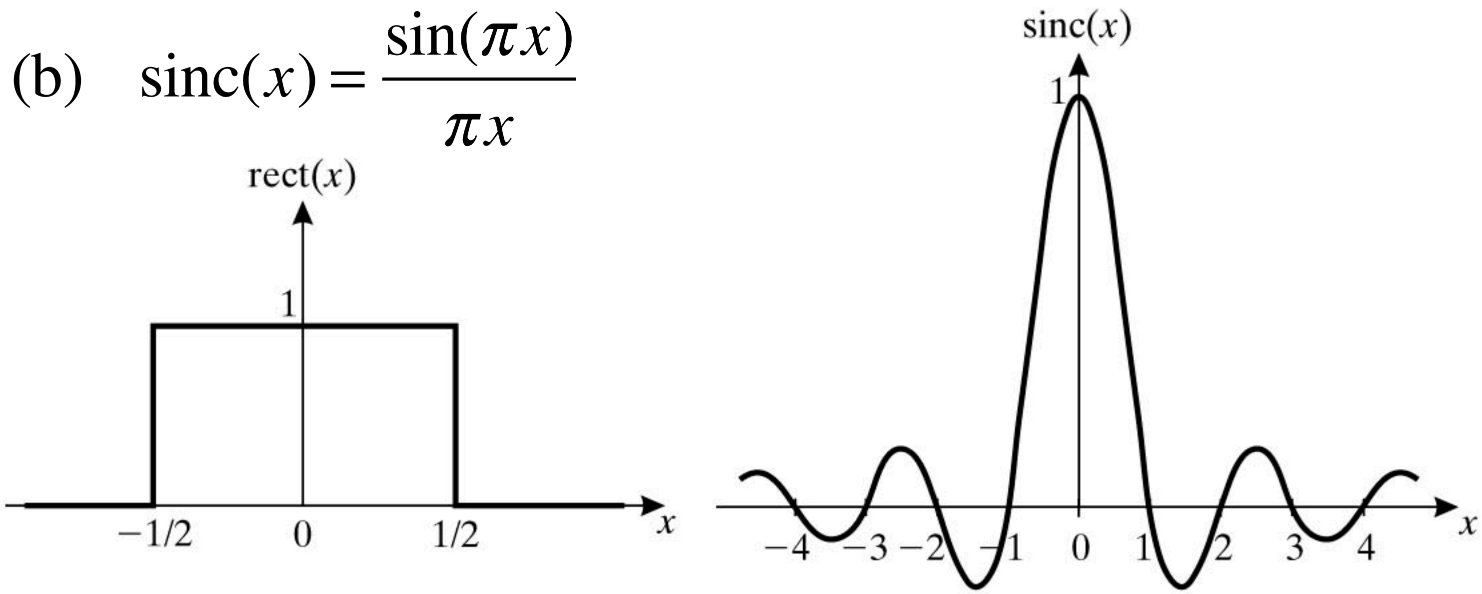
- For data measurements the element size is critical (e.g. Shannon's sampling theorem)
- For most of our work we will use continuous function theory for convenience, but sometimes the discrete theory will be required

Important signals - rect() and sinc() functions

- 1D rect() and sinc() functions
  - both have unit area

$$\text{(a)} \quad \mathrm{rect}(x) = \begin{cases} 1, & \text{for } |x| < 1/2 \\ 0, & \text{for } |x| > 1/2 \end{cases}$$

$$\text{(b)} \quad \mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



(a) (b)

Important signals - 2D rect() and sinc() functions

- 2D rect() and sinc() functions are straightforward generalizations

$$\text{(a)} \quad \operatorname{rect}(x,y) = \begin{cases} 1, & \text{for } |x| < 1/2 \text{ and } |y| < 1/2 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{(b)} \quad \operatorname{sinc}(x,y) = \frac{\sin(\pi x)\sin(\pi y)}{\pi^2 xy}$$

- *Try to sketch these*
- 3D versions exist and are sometimes used
- Fundamental connection between rect() and sinc() functions and very useful in signal and image processing

Important signals - Impulse function

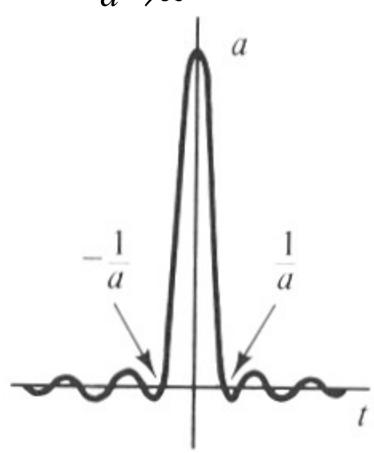
- 1D Impulse (delta) function
- A 'generalized function'
  - operates through integration
  - has zero width and unit area
  - has important 'sifting' property
  - can be understood by considering:

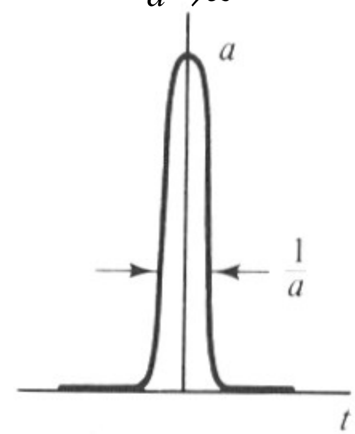
$$\begin{cases} \delta(x) = 0, \quad x \neq 0, \\ \int_{-\infty}^{\infty} \delta(x)\,dx = 1 \end{cases}$$

$$\begin{cases} \int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0) \\ \int_{-\infty}^{\infty} f(x)\delta(x-t)\,dx = f(t) \end{cases}$$

- Ways to approach the delta function

$$\delta(t) = \lim_{a\to\infty} a\,\mathrm{rect}(at) \qquad \delta(t) = \lim_{a\to\infty} a\,\mathrm{sinc}(at) \qquad \delta(t) = \lim_{a\to\infty} a e^{-\pi a^2 t^2}$$





Exponential and sinusoidal signals

- Recall Euler's formula,which connects trigonometric and complex exponential functions (*j* = sqrt(-1))

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

(not *i*)

- The exponential signal is defined as:

$$e^{j2\pi x} = \cos(2\pi x) + j\sin(2\pi x), \quad \text{where } j^2 = -1$$

- u_0 and v_0 are the fundamental frequencies in *x*- and *y*-directions, with units of 1/distance $e(x,y) = e^{j2\pi(u_0 x + v_0 y)}$

- We can write $e(x,y) = e^{j2\pi(u_0 x + v_0 y)}$

$$= \underbrace{\cos\left[2\pi\left(u_0 x + v_0 y\right)\right]}_{\text{real and even}} + \underbrace{j\sin\left[2\pi\left(u_0 x + v_0 y\right)\right]}_{\text{imaginary and odd}}$$

Exponential and sinusoidal signals

- Recall that $$\sin(2\pi x) = \frac{1}{2j}\left(e^{j2\pi x} - e^{-j2\pi x}\right)$$

$$\cos(2\pi x) = \frac{1}{2}\left(e^{j2\pi x} + e^{-j2\pi x}\right)$$

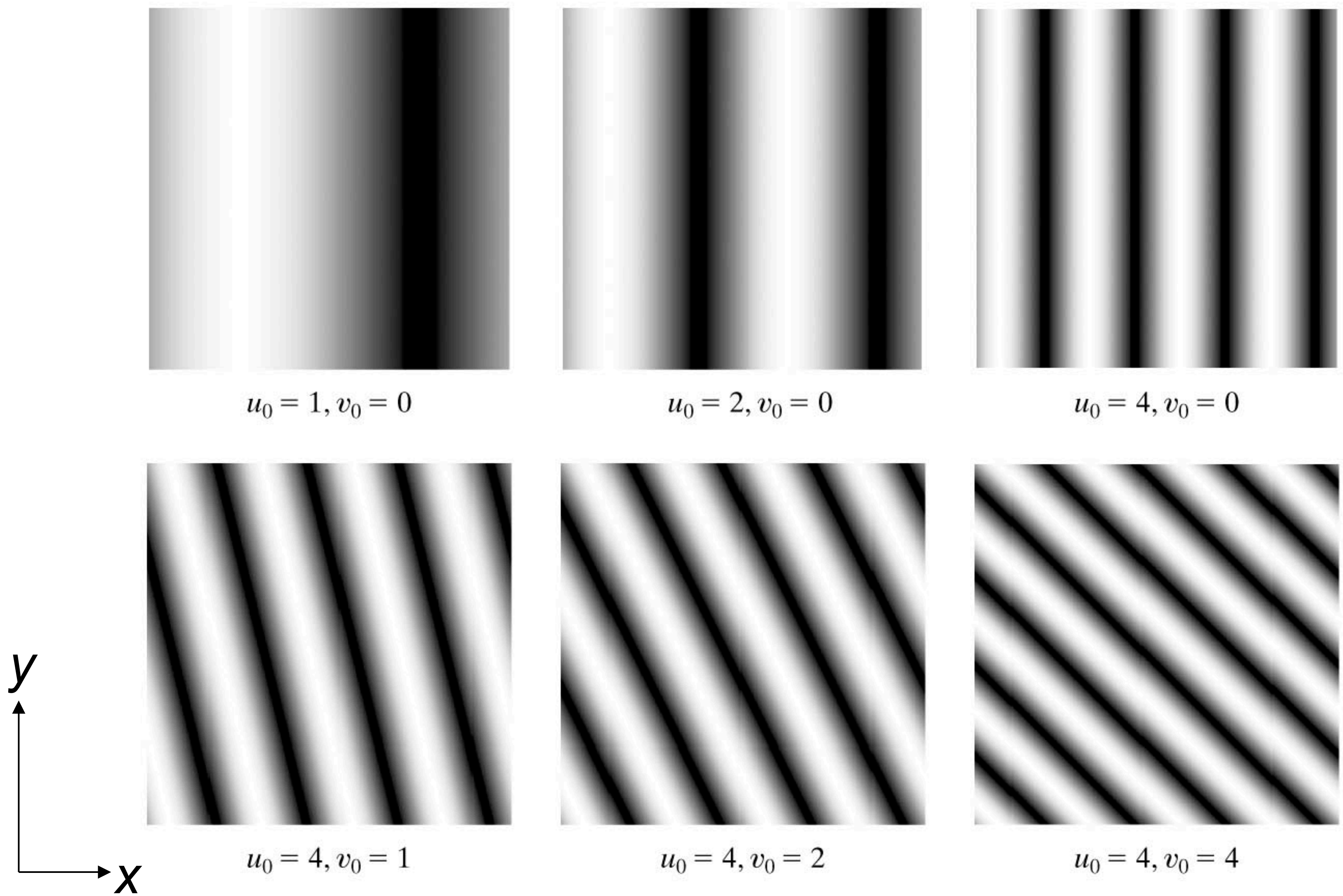
- so we have $$\sin\left[2\pi\left(u_0 x + v_0 y\right)\right] = \frac{1}{2j}\left(e^{j2\pi\left(u_0 x + v_0 y\right)} - e^{-j2\pi\left(u_0 x + v_0 y\right)}\right)$$

$$\cos\left[2\pi\left(u_0 x + v_0 y\right)\right] = \frac{1}{2}\left(e^{j2\pi\left(u_0 x + v_0 y\right)} + e^{-j2\pi\left(u_0 x + v_0 y\right)}\right)$$

- Fundamental frequencies u_0, v_0 affect the oscillations in x and y directions, E.g. small values of u_0 result in slow oscillations in the x-direction
- These are complex-valued and directional plane waves

Exponential and sinusoidal signals

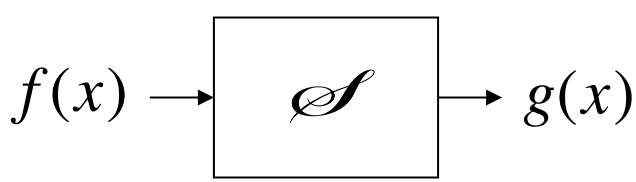
- Intensity images for $s(x,y) = \sin\left[2\pi\left(u_0 x + v_0 y\right)\right]$



System models

- Systems analysis is a powerful tool to characterize and control the behavior of biomedical imaging devices
- We will focus on the special class of *continuous*, *linear*, *shift-invariant* (LSI) systems
- Many (all) biomedical imaging systems are not really any of the three, but it can be useful tool, as long as we understand the errors in our approximation
- "all models are wrong, but some are useful" -*George E. P. Box*
- Continuous systems convert a continuous input to a continuous output
- Some notation examples:

$$g(x) = \mathscr{S}\left[f(x)\right] \quad \left(g(t) = \mathscr{S}\left[f(t)\right]\right)$$



Linear Systems

- A system $\mathscr{S}$ is a linear system if: we have $\mathscr{S}\left[f(x)\right]=g(x)$

  then $$\mathscr{S}\left[a_1 f_1(x)+a_2 f_2(x)\right]=a_1 g_1(x)+a_2 g_2(x)$$

or in general $$\mathscr{S}\left[\sum_{k=1}^{K} w_k f_k(x)\right]=\sum_{k=1}^{K} w_k\, \mathscr{S}\left[f_k(x)\right]=\sum_{k=1}^{K} w_k g_k(x)$$

- Is this a linear system? $g(x)=e^{\pi} f(x)$

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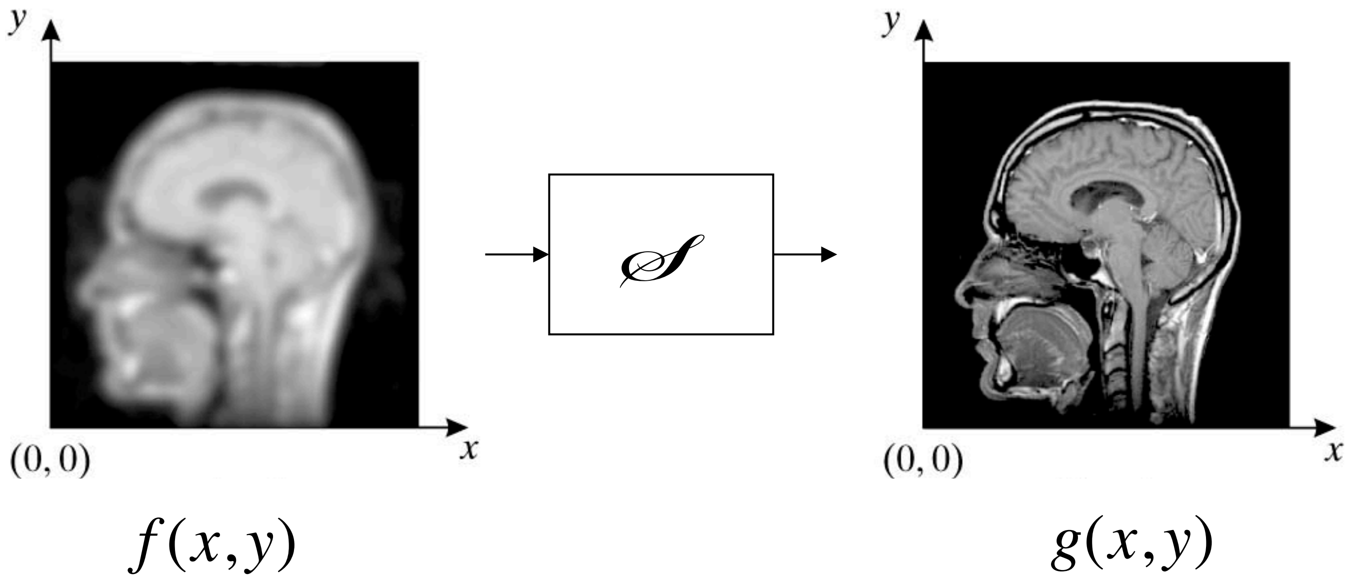
- Is this a linear system? $g(x)=e^{\pi} f(x)$

- Yes! $$\begin{aligned}\mathscr{S}\left[a_1 f_1(x)+a_2 f_2(x)\right]&=e^{\pi}\left(a_1 f_1(x)+a_2 f_2(x)\right)\\&=a_1 e^{\pi} f_1(x)+a_2 e^{\pi} f_2(x)\\&=a_1 g_1(x)+a_2 g_2(x)\end{aligned}$$

2D Linear Systems

- Now use 2D notation
- Example: sharpening filter



- In general

$$\mathscr{S}\left[\sum_{k=1}^{K} w_k f_k(x,y)\right] = \sum_{k=1}^{K} w_k \, \mathscr{S}\left[f_k(x,y)\right] = \sum_{k=1}^{K} w_k g_k(x,y)$$

Shift-Invariant Systems

- Start by shifting the input $f_{x_0 y_0}(x,y) \triangleq f(x-x_0, y-y_0)$

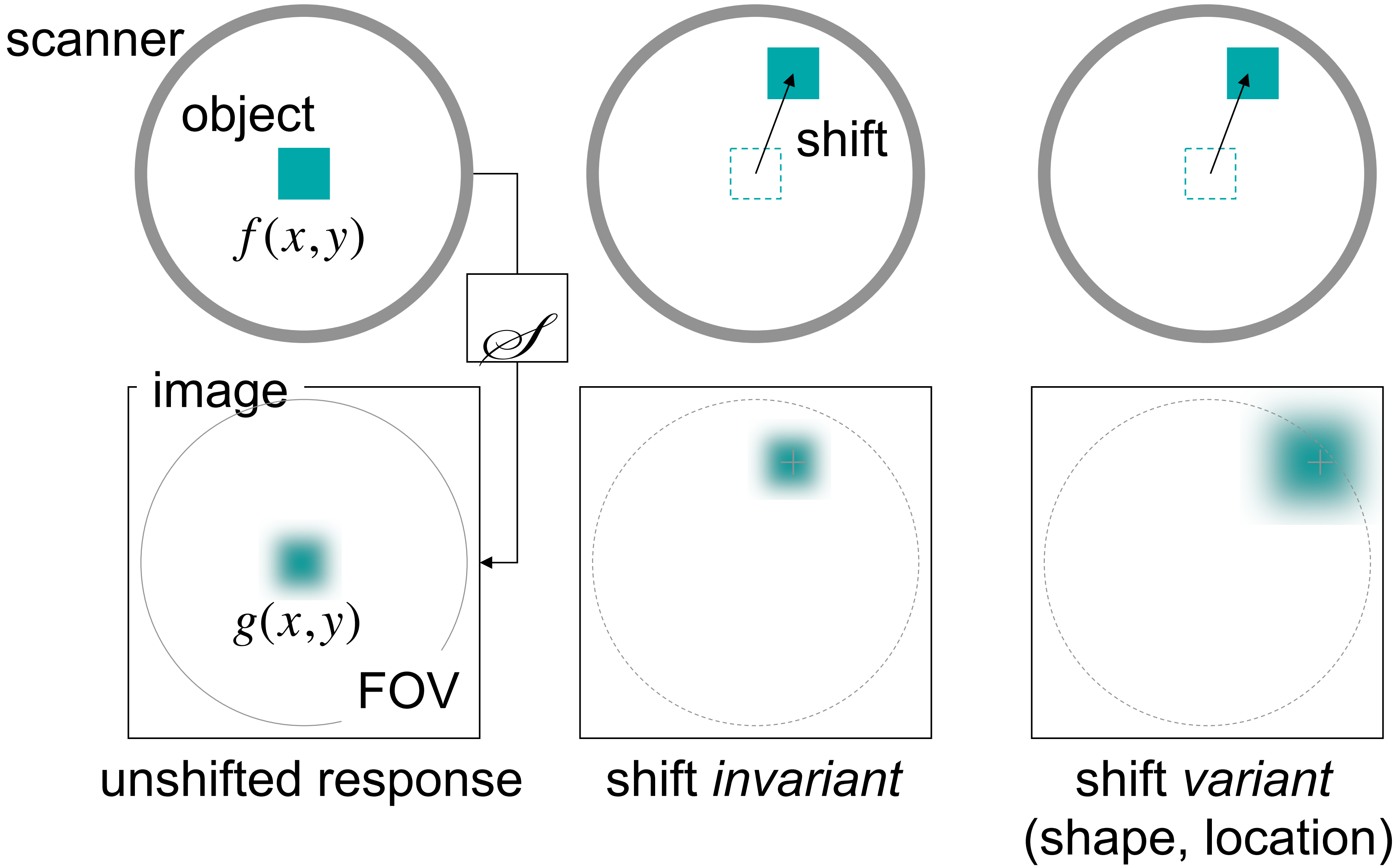
  then if

  $$g_{x_0 y_0}(x,y) = \mathscr{S}\left[f_{x_0 y_0}(x,y) \right] = g(x-x_0, y-y_0)$$

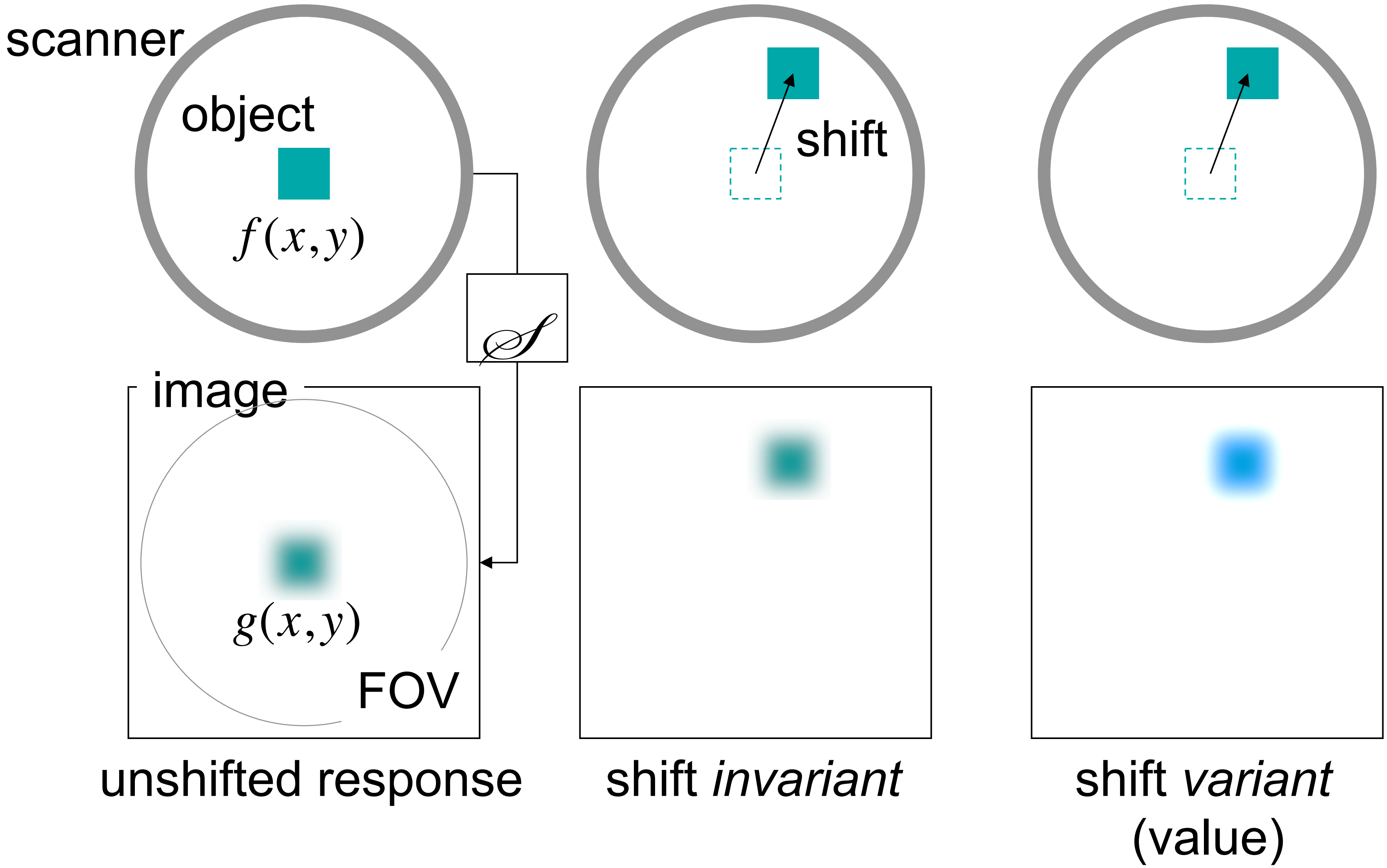
  the system is *shift-invariant*, i.e. response does not depend on location

- Shift-invariance is separate from linearity, a system can be
  - shift-invariant and linear
  - shift-invariant and non-linear
  - shift-variant and linear
  - shift-variant and non-linear

Shift invariant and shift-variant system response

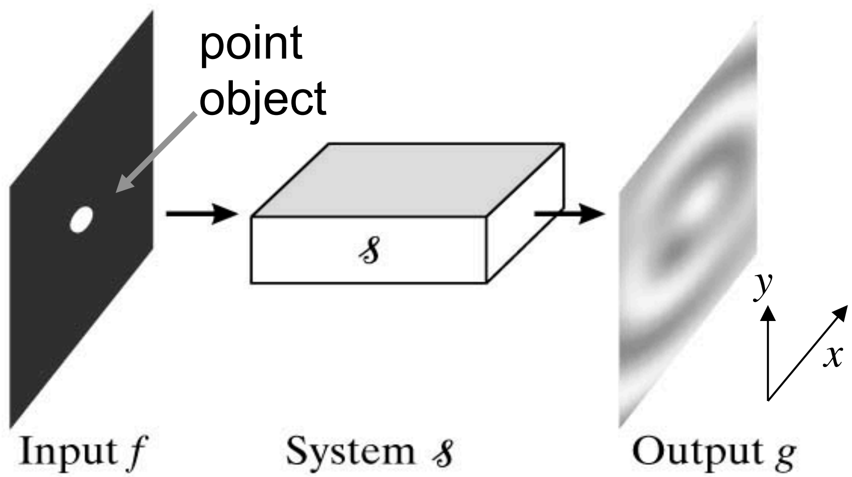


Shift invariant and shift-variant system response



Impulse Response

- Linear, shift-invariant (LSI) systems are the most useful
- First we start by looking at the response of a system using a point source at location (ξ,η) as an input



input $f_{\xi\eta}(x,y) \triangleq \delta(x-\xi, y-\eta)$

output $g_{\xi\eta}(x,y) \triangleq h(x,y;\xi,\eta)$

- The output *h()* depends on location of the point source (ξ,η) and location in the image (*x,y*), so it is a 4-D function
- Since the input is an impulse, the output is called the *impulse response function,* or the *point spread function* (PSF) - *why?*

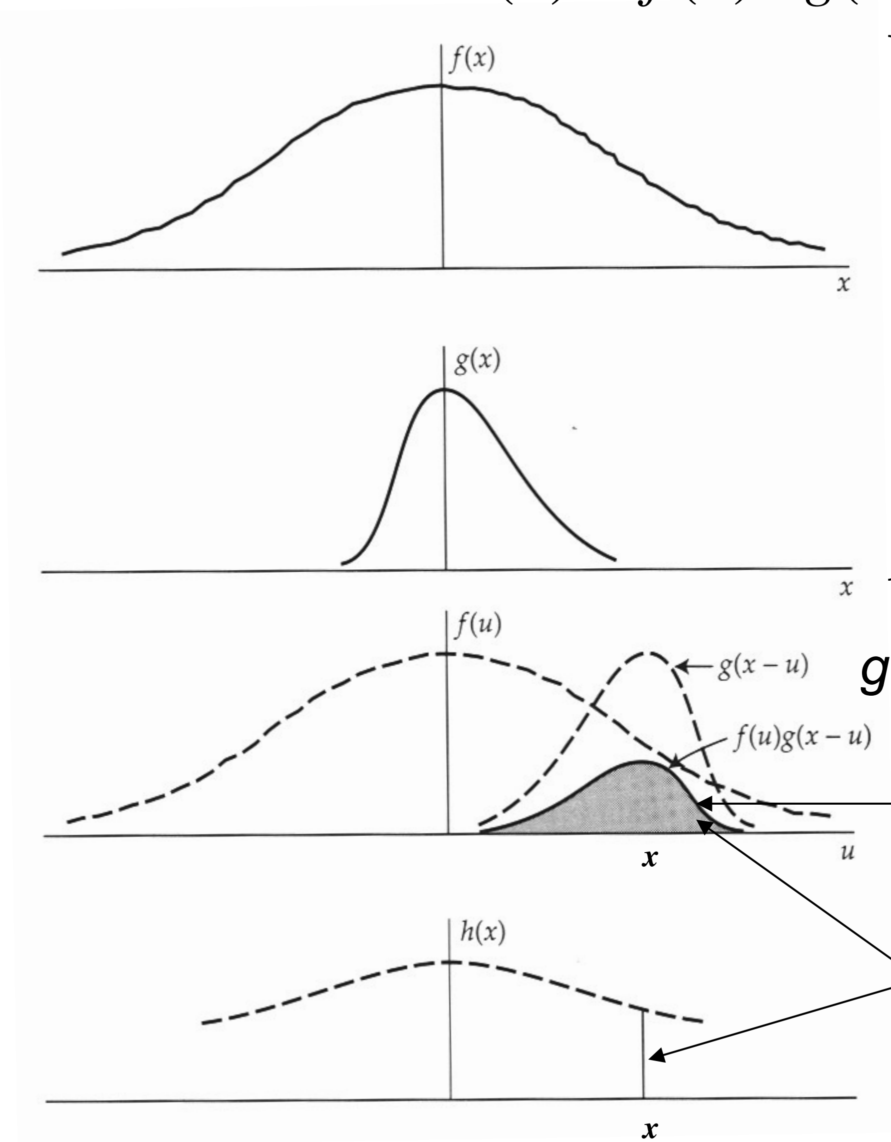
Impulse Response of Linear Shift Invariant Systems

- For LSI systems
$$\mathscr{S}\left[f(x-x_0,y-y_0)\right]=g(x-x_0,y-y_0)$$
- So the PSF is
$$\mathscr{S}\left[\delta(x-x_0,y-y_0)\right]=h(x-x_0,y-y_0)$$
- Through something called the superposition integral, we can show that
$$g(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\xi,\eta)\,h(x,y;\xi,\eta)\,d\xi\,d\eta$$
- And for LSI systems, this simplifies to:
$$g(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\xi,\eta)\,h(\xi-x,\eta-y)\,d\xi\,d\eta$$
- The last integral is a convolution integral, and can be written as

$$g(x,y)=f(x,y)*h(x,y)\quad(\text{or}\,f(x,y)**h(x,y))$$

Review of convolution

- Illustration of $h(x) = f(x) * g(x) = \int_{-\infty}^{\infty} f(u)g(x-u)\,du$



original functions

g(x-u), reversed and shifted to *x*

curve = product of *f(u)g(x-u)*

area = integral of *f(u)g(x-u)*
= value of *h()* at *x*